

# TQFT and Whitehead's manifold

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## Abstract

The aim of this note is to derive some invariants at infinity for open 3-manifolds in the framework of Topological Quantum Field Theories. These invariants may be used to test if an open manifold is simply connected at infinity as we done for Whitehead's manifold in case of the  $sl_2(\mathbf{C})$ -TQFT in level 4.

## 1 Introduction

The aim of this paper is to introduce the invariants at infinity for open 3-manifolds deduced from Topological Quantum Field Theories (abbrev. TQFT). We are able to compute the invariant of the classical Whitehead manifold associated to the simplest non-abelian TQFT based on  $sl_2(\mathbf{C})$  in level 4 and find it is not trivial. As a consequence the manifold is not simply connected at infinity. Further developpements accreditating the idea that TQFT may share some light on the topology of open 3-manifold are pursued in a further paper.

In the first section we discuss the general TQFT invariants at infinity  $Z_\infty(W)$  for an open 3-manifold  $W$ . In order to preserve the self-contained character of this paper we outline the definition of link and 3-manifold invariants of Witten and Reshetikhin-Turaev [?, ?] based on the quantum  $sl_2(\mathbf{C})$  at roots of unity. According to a general result each multiplicative invariant for closed 3-manifolds extends canonically to a TQFT ([?]). We follow the surgical approach used by Turaev in [?] to get the explicit description of the TQFT for cobordisms.

In the last section we use these results to compute effectively the TQFT invariant at infinity for the Whitehead manifold. Recall ([?]) this is defined as follows: Let  $T_1 \hookrightarrow T_0$  be the embedding of the solid tori from picture 1. There exists an homeomorphism  $h$  of  $S^3$  so that  $h(T_1) = T_0$ . Consider the open manifold  $Wh = \bigcup_{n \geq 0} h^n(T_0)$ . Then  $Wh$  is the typical example of a contractible open 3-manifold which is not homeomorphic to  $\mathbf{R}^3$ . The precise reason is that

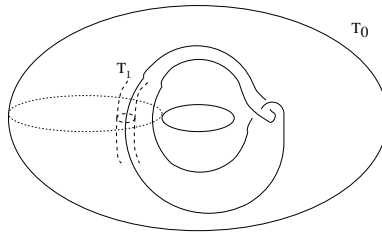


Figure 1: The inclusion of tori

$Wh$  is not simply connected at infinity (i.e. not every compact may be engulfed in a compact simply-connected submanifold).

Our main result states as follows:

**Theorem 1.1** *The space  $Z_\infty(W) \cong \mathbf{C}^3$  if  $Z$  is the  $sl_2(\mathbf{C})$ -TQFT at level 4.*

Since any open 3-manifold  $W$  which is simply-connected at infinity must satisfy  $\dim Z_\infty(W) \leq 1$  for all reduced TQFT (see Proposition 2.2) we get another proof for the non simple connectedness at infinity of  $Wh$ . It seems that  $Z_\infty$  is a good test for the simple connectedness at infinity even if not enough strong to distinguish among various non-homeomorphic open 3-manifolds.

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## 2 Invariants at infinity from TQFT

(2.1) **On TQFT.** Recall [?] that a TQFT in dimension 3 is a functor  $Z$  from the category of oriented cobordisms into that of hermitian vector spaces. This means that to a compact surface  $S$  we associate an hermitian vector space  $Z(S)$  depending only on the topological type of  $S$ . The *quantum* character of the theory is reflected in the rules

$$Z(\cup_i S_i) = \otimes_i Z(S_i), \quad Z(\emptyset) = \mathbf{C},$$

which make the difference with the usual functors encountered in the algebraic topology.

Furthermore to an oriented cobordism  $M$  so that  $\partial M$  is split into two disjoint manifold  $S$  and  $T$  (the *incoming* and the *outgoing* boundaries respectively, which are not necessary those given by the orientation) we have assigned a morphism  $Z(M) : Z(S) \longrightarrow Z(T)$  satisfying the natural compatibility relations between composition of morphisms and cobordisms. This is usually called an anomaly-free TQFT (see [?]). The main examples yet constructed have an anomaly from a certain group of roots of unity  $\Gamma \subset U(1)$ . This means that the invariant associated to the composition of cobordisms  $M$  and  $N$  may be expressed as

$$Z(M \circ N) = \gamma Z(M) \circ Z(N), \quad \text{with } \gamma \in \Gamma.$$

The usual way to deal with this ambiguity is to work with framed 3-manifolds [?] or  $p_1$ -structures [?]. However the presence of an anomaly will be irrelevant for the construction of invariants at infinity.

The examples we consider in this paper are *reduced* TQFT, namely they satisfy the additional condition

$$Z(S^2) \cong \mathbf{C}.$$

All the TQFT from quantum groups or quasi-quantum groups are reduced and we may restrict ourselves to the study of reduced TQFT by the results of [?].

(2.2) **Open 3-manifolds.** Consider first  $Z$  is an anomaly-free TQFT. Let  $W$  be an open 3-manifold without boundary. We choose an ascending sequence of submanifolds  $\{K_n\}$  fulfilling

$$K_n \subset \text{int}(K_{n+1}), \quad W = \cup_n K_n.$$

Then  $V_i = \text{cl}(K_{i+1} - K_i)$  are oriented cobordism from  $\partial K_i$  to  $\partial K_{i+1}$ . Here *int* and *cl* state for the interior and the closure respectively. We get a sequence of linear maps

$$Z(V_i) : Z(\partial K_i) \longrightarrow Z(\partial K_{i+1}),$$

which represent an inductive system of vector spaces. We define  $Z_\infty(W)$  be simply the inductive limit of this system.

**Definition-Lemma 2.1** *The vector space  $Z_\infty(W)$  is the topological invariant at infinity associated to the TQFT functor  $Z$  and the open 3-manifold  $W$ .*

In fact it is simply to check the independence of  $Z_\infty(W)$  on the choice of the exhaustion or the parametrizations of the intermediary boundaries  $\partial K_i$ .  $\square$

Remark that  $Z_\infty(W)$  depends only on the structure at infinity of  $W$ : if  $W'$  is another manifold so that  $W$  and  $W'$  are homeomorphic outside some compacts then the associated spaces  $Z_\infty(W)$  and  $Z_\infty(W')$  are isomorphic.

Also if  $Z$  is a TQFT with anomaly this time, then the maps  $Z(V_i)$  are defined up to the multiplication by some scalar from  $\Gamma$ . Nevertheless the space  $Z_\infty(W)$  is well determined. It is only the hermitian structure which is lost when we pass to the limit  $Z_\infty(W)$ .

(2.3)  **$h$ -1-connected manifolds.** The open 3-manifold  $W$  is  *$h$ -1-connected at infinity* if each compact  $K \subset W$  may be engulfed in a compact submanifold  $Y \subset W$  with  $H_1(Y) = 0$ . This is a condition slightly weaker than the simple connectedness at infinity.

**Proposition 2.2** *If  $W$  is  $h$ -1-connected at infinity then  $\dim Z_\infty(W) \leq 1$  for any reduced TQFT.*

Proof: It suffices to observe that a compact 3-manifold  $Y$  with  $H_1(Y) = 0$  has the boundary  $\partial Y$  an union of spheres  $S^2$ , from an Euler characteristic argument. If  $\{K_n\}$  is an exhaustion of  $W$  like in the introduction then there exists compact submanifold  $Y_n$  with  $H_1(Y_n) = 0$  and a function  $r(n) > n$  so that

$$K_n \subset \text{int}(Y_n) \text{ and } Y_n \subset \text{int}(K_{r(n)}).$$

Then the map  $Z(\text{cl}(K_{r(n)} - K_n))$  factors through  $Z(\partial Y_n) \cong \mathbf{C}$  (because the TQFT is reduced) hence the rank of the limit is at most 1.  $\square$

(2.4) **The Whitehead manifold.** Consider now  $Z$  be the level 4  $sl_2(\mathbf{C})$ -TQFT of Witten and Reshetikhin-Turaev (see the next section for complete definitions). The main result of this paper stated in introduction asserts that  $Z_\infty(Wh) \cong \mathbf{C}^3$ .

We defer for the proof in section 4. We already notice that  $Wh$  has periodic ends and we may choose  $K_n = \bigcup_{0 \leq j \leq n} h^j(T_0)$ . Then all intermediary cobordisms  $V_n$  are homeomorphic to  $X = \text{cl}(T_0 - T_1)$ . It suffices therefore to compute the linear map  $Z(X) : Z(\partial T_1) \longrightarrow Z(\partial T_0)$ . For the TQFT we are working with, the space associated to a torus  $Z(S^1 \times S^1)$  is  $\mathbf{C}^3$  (see the further section). So the statement of the theorem is equivalent to the non-degeneracy of the linear map  $Z(X)$ .

### 3 The $sl_2(\mathbf{C})$ -TQFT

We fix some integer  $r > 1$  called the level of the theory. The description we outline follows from [?, ?, ?].

(3.1) **Framed tangles.** Recall that a tangle  $T$  is a 1-manifold properly embedded in the unit cube  $I^3$  in  $\mathbf{R}^3$  with  $\partial T \subset \{\frac{1}{2}\} \times I \times \partial I$ , considered up to isotopy rel boundary. If  $\partial_- T = T \cap I^2 \times \{0\}$  and  $\partial_+ T = T \cap I^2 \times \{1\}$  then  $T$  is a  $(m, n)$ -tangle provided that  $|\partial_- T| = m$  and  $|\partial_+ T| = n$ . Thus a link is a  $(0, 0)$ -tangle and a general tangle consists of a link with a collection of proper arcs. We assume the tangles are oriented and transverse to  $I^2 \times \partial I$ .

A *framed tangle* is a tangle equipped with a framing of its normal bundle (up to isotopy) which is standard on the boundary. It is equivalent to a ribbon tangle from [?] if we think the tangle is thickened to a ribbon in the direction of the second vector of the framing. Anyway framings may be specified by integers assigned to the components of  $T$ .

Also one studies tangles using generic projections, called *diagrams*, onto  $\{0\} \times I^2$  having only ordinary double points. We assume the framing considered is the blackboard one, in which the second vector is parallel to  $\{0\} \times I^2$ , and further coincides with the 0-framing by eventually adding the necessary number of kinks.

It is simply to check that every tangle diagram may be factored into the elementary tangles from picture 2. There are well-known Reidemeister moves describing the local moves necessary

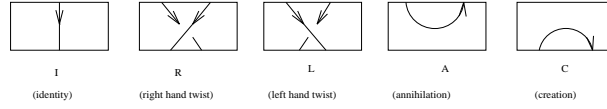


Figure 2: Elementary tangles

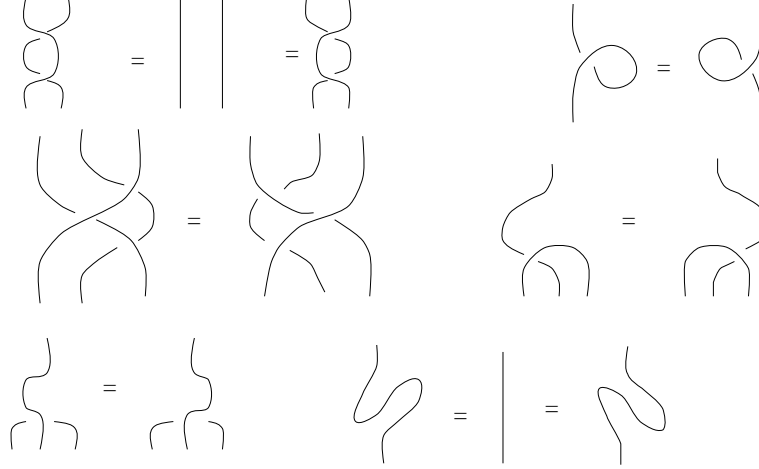


Figure 3: Reidemester moves for tangles

and sufficient to obtain two framed tangle diagrams one from the other if they are coming from the same framed tangle. For the sake of completeness we pictured them in figure 3 (see [?]). Notice the orientations are arbitrary and the framing is the blackboard one.

(3.2) **Quasi-Triangular Hopf algebras.** We discuss the quantum  $sl_2(\mathbf{C})$  which is the main example of a ribbon Hopf algebra. Recall  $sl_2(\mathbf{C})$  is 3-dimensional as vector space and the Lie bracket is given (in terms of preferred generators) by:

$$[H, X] = 2X, [H, Y] = -2Y, [X, Y] = H.$$

The *universal enveloping algebra*  $U = U(sl_2(\mathbf{C}))$  is the associative algebra over  $\mathbf{C}$  generated by  $X, Y, H$  and the relations from above. Notice that there exists an unique  $k$ -dimensional irreducible  $sl_2(\mathbf{C})$ -module  $V^k$ , for each integer  $k$ , which has also an  $U$ -module structure. Also  $U$  is a Hopf algebra when endowed with the comultiplication  $\Delta : U \longrightarrow U \otimes U$  given by  $\Delta(u) = u \otimes 1 + 1 \otimes u$ , antipode  $s : U \longrightarrow U$  given by  $s(u) = -u$ , and counit  $\varepsilon : U \longrightarrow \mathbf{C}$  determined by  $\varepsilon(u) = 0$ , for all  $u$  which are Lie polynomials in  $X, Y, H$ .

Now the *quantized universal enveloping algebra*  $U_h = U_h(sl_2\mathbf{C})$  is defined as  $U[[h]]$  (the formal series in  $h$ ) with the same relations as  $U$  excepting for  $[X, Y] = H$  which is replaced by

$$[X, Y] = [H] = \frac{e^{\frac{h}{2}H} - e^{-\frac{h}{2}H}}{e^{\frac{h}{2}} - e^{-\frac{h}{2}}} \quad (1)$$

If  $K = e^{\frac{h}{4}H}$  we have the relations

$$\begin{aligned} KX &= e^{\frac{h}{2}} XK \\ KY &= e^{-\frac{h}{2}} YK \end{aligned}$$

$$[X, Y] = \frac{K^2 - K^{-2}}{e^{\frac{h}{2}} - e^{-\frac{h}{2}}}. \quad (2)$$

Notice that there is a Hopf algebra structure on  $U_h$  as a module over  $\mathbf{C}[[h]]$ .

Following [?] we consider  $A$  be the quotient of  $U_h$  obtained by setting

$$h = \frac{2\pi\sqrt{-1}}{r}, \quad X^r = Y^r = 0, \quad K^{4r} = 1.$$

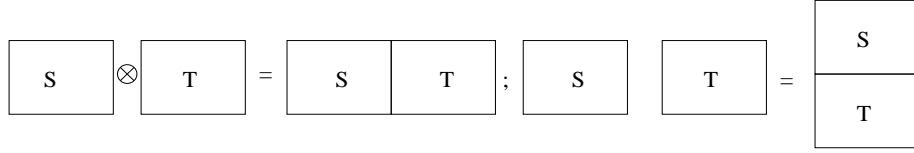


Figure 4: Composition laws for tangles

Then  $A$  is a finite dimensional algebra over the complex numbers with generators  $X, Y, K, K^{-1}$  and the relations stated above. As in the case of  $U$  there are unique  $A$ -modules  $V^k$  in each dimension  $k$  but  $V^k$  is irreducible only if  $k \leq r$ . Also  $A$  acquires a Hopf algebra structure from  $U_h$  and so tensor products and duals of  $A$ -modules are still  $A$ -modules. Moreover the following Clebsch-Gordon rules remain valid

$$V^k \otimes V^l = \bigoplus_{p=|l-k|+1; p+k+l=\text{odd}} V^p, \text{ if } k+l \leq r+1 \quad (3)$$

This  $A$  is a *quasi-triangular* Hopf algebra (see [?]): there exists an invertible element  $R \in A \otimes A$  satisfying

$$\begin{aligned} R\Delta(u)R^{-1} &= \check{\Delta}(u), u \in A, \\ (\Delta \otimes 1)(R) &= R_{13}R_{23}, \\ (1 \otimes \Delta)(R) &= R_{13}R_{12}, \end{aligned}$$

where  $\check{\Delta} = P\Delta$ ,  $P$  is the permutation endomorphism of  $A \otimes A$ ,  $P(u \otimes v) = v \otimes u$ ,  $R_{12} = R \otimes 1$ ,  $R_{23} = 1 \otimes R$ ,  $R_{13} = (P \otimes 1)R_{23}$ .

Specifically  $R$  may be given by

$$R = \frac{1}{4r} \sum_{0 \leq n, a, b}^{n < r; a, b < 4r} \frac{e^{\frac{h}{2}} - e^{-\frac{h}{2}}}{[n!]} t^{ab+(b-a+1)n} (X^n K^a + Y^n K^b) \quad (4)$$

where  $t = e^{-\frac{2\pi\sqrt{-1}}{4r}}$ .

Remark that  $R$  may be viewed as acting on tensor products of two  $A$ -modules  $V \otimes W$ . We set  $\check{R} : V \otimes W \longrightarrow W \otimes V$  be the *flip*  $R$ -matrix  $\check{R} = P \circ R$ .

**(3.3) Colored framed tangle operators.** Assume the quasi-triangular Hopf algebra  $A$  is fixed. A coloring of a tangle  $T$  is the assignment of an  $A$ -module to each of its components. This way a coloring of  $\partial T$  is induced: if  $s$  is an arc colored by  $V$  then assign  $V$  to the endpoint of  $s$  where is oriented down, and the module  $V^*$  to the other one. Tensoring from left to the right the modules associated to the bottom (or upper) endpoints we get the boundary  $A$ -modules assigned to  $\partial_- T$  and  $\partial_+ T$ , which we denote  $T_-$  and  $T_+$  respectively. By convention the empty product is  $\mathbf{C}$ .

We have two composition laws on tangles  $\circ$  and  $\otimes$  illustrated in picture 4.

**Theorem 3.1** ([?, ?]) *There exist uniquely  $A$ -linear operators  $J_T : T_- \longrightarrow T_+$  assigned to each colored tangle which satisfy*

$$J_{S \circ T} = J_S \circ J_T,$$

$$J_{S \otimes T} = J_S \otimes J_T,$$

and for elementary tangles are defined by

$$J_I = 1, J_R = \check{R}, J_L = \check{R}^{-1}, J_{CR} = E, J_{CL} = \check{E}, J_{AR} = N, J_{AL} = \check{N},$$

where  $R$  is the right hand twist (the orientation points down) tangle,  $L$  is the left hand twist,  $CR$  (respectively  $AR$ ) is the creation (annihilation) tangle with the sense of the orientation from left to the right,  $CL$  (respectively  $AL$ ) have opposite orientation than  $CR$  and  $AR$  respectively,

$$E(f \otimes x) = f(x), \check{E}(x \otimes f) = f(K^2 x),$$

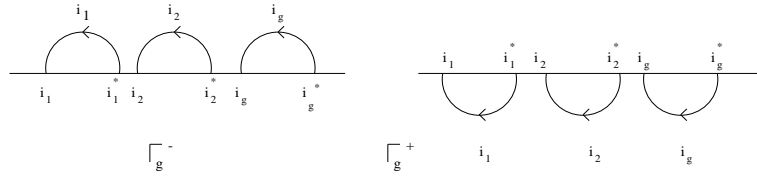


Figure 5: The standard spine for  $\Sigma_g$

$$\begin{aligned} N(1) &= \sum_i e_i \otimes e^i, \{e_i\} \text{ is an arbitrary basis and } \{e^i\} \text{ its dual,} \\ \check{N}(1) &= \sum_i e^i \otimes K^{-2}e_i. \end{aligned}$$

Notice that  $J_K$  is just a scalar if  $K$  is a colored link.

Now we restrict ourselves to colorings by irreducible  $A$ -modules so the set of colors correspond to  $\{1, 2, \dots, r\}$ . We denote by  $\mathbf{k}$  the coloring of  $T$  where the  $j^{\text{th}}$  component is colored with the module of dimension  $k_j$ . Thus the theorem yields a family of topological invariants  $J_{T, \mathbf{k}}$  for colored tangles.

Remark that  $J_{T, \mathbf{k}}$  are independent on the various orientations of closed components of  $T$ . Also from ([?] p.506) if a color in the vector  $\mathbf{k}$  is  $r$  then the invariant  $J_{T, \mathbf{k}} = 0$ . So we may assume the colors are from the subset  $\{1, 2, \dots, r-1\}$ .

**(3.4) Closed 3-manifold invariants.** Let  $L$  be a framed link in  $S^3$ . Recall  $L$  determines a 4-manifold  $W_L$  obtained by adding 2-handles to the 4-ball  $B^4$  along the components of  $L$  in  $S^3 = \partial B^4$ . The manifold  $D(L) = \partial W_L$  oriented "outward first" is the result of Dehn surgery on  $L$ , and any 3-manifold may be obtained this way. We can pass from one surgery link  $L$  for  $M = D(L)$  to another link  $L'$  with  $D(L') = M$  by a finite sequence of Kirby moves (blow-ups and handle slidings) or equivalently  $m$ -strands K-moves (see [?]).

Define for a framed link  $L$

$$Z_L = \alpha_L \sum_{\text{coloring } \mathbf{k}} [k] J_{L, \mathbf{k}}, \quad (5)$$

where  $\alpha_L = b^{n_L} c^{\sigma(L)}$ ,  $[\mathbf{k}] = \prod [k_i]$ ,  $[k] = \frac{e^{\frac{kh}{2}} - e^{-\frac{kh}{2}}}{e^{\frac{h}{2}} - e^{-\frac{h}{2}}} = \frac{\sin \frac{\pi k}{r}}{\sin \frac{\pi}{r}}$ ,  $b = \sqrt{\frac{2}{r}} \sin \frac{\pi}{r}$ ,  $c = e^{-\frac{6\pi\sqrt{-1}(r-2)}{8r}}$ ,  $n_L$  is the number of components of  $L$ ,  $\sigma_L$  is the signature of the linking matrix of  $L$ .

The invariance of  $Z_L$  to Kirby moves is proved in [?]. Now if  $M$  is obtained by Dehn surgery on the framed link  $L$  we set  $Z_r(M) = Z_L \in \mathbf{C}$  which is a topological invariant for 3-manifolds.

**(3.5) Cobordisms and TQFT.** For simplicity we restrict ourselves to cobordisms  $M$  with connected boundaries  $\partial_- M$  (incoming) and  $\partial_+ M$  (outgoing).

We define first the spaces associated to closed oriented surfaces. Consider  $(\Gamma, \mathbf{i})$  be one of the colored 3-valent graph of genus  $g$  from picture 5, viewed as a  $(0, 2g)$ -tangle in  $\mathbf{R}^3$ .

We associate to this colored graph the space

$$Z(\Gamma_g, \mathbf{i}) = \bigotimes_{i=1}^g (V^{i_i} \otimes V^{i_i^*}).$$

Then to the closed oriented surface of genus  $g$  we assign the space

$$Z(\Sigma_g) = \bigoplus_{\text{coloring } \mathbf{k}} Z(\Gamma_g, \mathbf{k}).$$

We may extend now the Dehn surgery construction to cobordisms using 3-valent graphs. We call  $\Gamma$  a special framed graph if it satisfies the conditions:

- (i)  $\Gamma \cap I^2 \times [0, \frac{1}{10}]$  is the base of a standard 3-valent graph  $\Gamma_{g_+}^+$ . This means that  $\Gamma \cap I^2 \times [0, \frac{1}{10}]$  from which the components not touching  $I^2 \times \{0\}$  are removed is isomorphic to  $\Gamma_{g_+}^+$ .
- (ii)  $\Gamma \cap I^2 \times [\frac{9}{10}, 1]$  is the base of the standard 3-valent graph  $\Gamma_{g_-}^-$ .
- (iii)  $\Gamma \subset I^3$  and the union of the components which do not touch the boundaries form a link  $L$ .

Now each special framed link  $\Gamma$  gives rise to a decorated cobordism  $(M, \Sigma_-, \Sigma_+)$  with parametrized surfaces  $\Sigma_-, \Sigma_+$  of genera  $g_-, g_+$  respectively. The construction goes as follows:

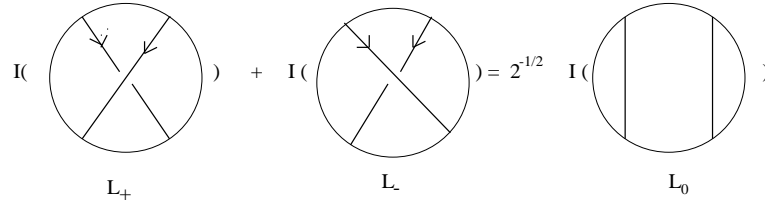


Figure 6: The skein relation

we have a regular neighborhood  $N(\Gamma_{g_+}^+) \subset S^3$  and an homeomorphism  $f^+ : H_g \longrightarrow N(\Gamma_{g_+}^+)$  from the handlebody of genus  $g$  (and a similar situation for  $\Gamma_{g_-}^-$ ). Cut out open handlebodies  $N(\Gamma_{g_+}^+)$  and  $N(\Gamma_{g_-}^-)$  from  $S^3$  to get a compact oriented 3-dimensional cobordism  $E$  between the respective surfaces. Now the maps  $f^+, f^-$  induce parametrizations of the boundary of  $E$ . Then surgery on  $E$  on the remaining link  $L$  produces a compact cobordisms  $M = D(\Gamma)$  with parametrized boundaries  $\partial_- M \cong \Sigma_{g_-}$  and  $\partial_+ M \cong \Sigma_{g_+}$ . Again each cobordism  $M$  whose boundary is partitioned into two disjoint parts may be obtained this way, and there are generalized Kirby moves for such surgery presentations (see [?] p.168, [?]).

We set now

$$Z_{\mathbf{j}}^{\mathbf{i}}(\Gamma) = b^{n_L - g_+} c^{\sigma(L)} [j] \sum_{\text{coloring } \mathbf{k}} [\mathbf{k}] J_{\Gamma, (\mathbf{i}, \mathbf{j}, \mathbf{k})} \quad (6)$$

where  $\mathbf{i}, \mathbf{j}$  are colorings of  $\Gamma_{g_-}^-, \Gamma_{g_+}^+$  respectively,  $\mathbf{k}$  is coloring of  $L$ , and  $(\Gamma, \mathbf{i}, \mathbf{j}, \mathbf{k})$  is viewed as a colored tangle giving rise to a map  $J_{\Gamma, \mathbf{i}, \mathbf{j}, \mathbf{k}} : Z(\Gamma_{g_-}^-, \mathbf{i}) \longrightarrow Z(\Gamma_{g_+}^+, \mathbf{j})$ . We set  $Z(\Gamma) : Z(\Sigma_{g_-}) \longrightarrow Z(\Sigma_{g_+})$  for the linear map whose blocks are the matrices  $Z_{\mathbf{j}}^{\mathbf{i}}(\Gamma)$ .

As  $Z(\Gamma)$  is invaried by Kirby moves it follows that the formula  $Z_r(M) = Z(\Gamma)$  defines a topological invariant for 3-dimensional oriented cobordisms with parametrized boundaries (i.e. decorated cobordisms).

Moreover we have

$$Z_r(M \circ N) = c Z_r(M) \circ Z_r(N),$$

where  $c$  lies in the group of roots of unity generated by  $e^{\frac{2\pi\sqrt{-1}}{r+2}}$ . Therefore this data is a TQFT with anomaly which we call the  $sl_2(\mathbf{C})$ -TQFT at level  $r$ .

## 4 The proof of the theorem

(4.1) **The Arf invariant.** We have a recurrent method to compute the Arf invariant (see [?]) of a proper link due to Murakami [?]. This is related to Jones polynomial at 4-th roots of unity. Specifically let  $I$  denote the link invariant defined by

$$I(\text{unknot}) = 1$$

$$I(\text{unknot} \cup K) = \sqrt{2} I(K), \text{ for any link } K,$$

and the skein relation

$$I(L_+) + I(L_-) = \sqrt{2} I(L_0),$$

where  $L_+, L_-$  are the left and right hand twists and  $L_0$  is the 2-parallel string diagram, and the rest of the diagrams are the same (see picture 6).

Then for a link  $L$  we have

$$I(L) = \begin{cases} (-1)^\varepsilon \sqrt{2^{n_L-1}} & \text{if } \text{Arf}(L) = \varepsilon \\ 0 & \text{if } L \text{ is not proper} \end{cases}$$

Remember that the link  $L$  is proper if, for each sub-link  $K$  the linking number  $lk(K, L - K)$  is even.

(4.2) **Jones polynomial at 4th roots of unity.** Consider now  $J_L = J_{L, \mathbf{2}}$ , where  $\mathbf{2}$  is the coloring of all components of  $L$  by the module  $V^2$ . Then  $J_L$  is a variant of Jones polynomial

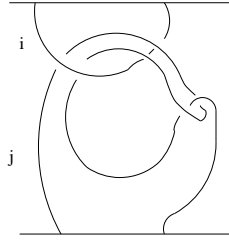


Figure 7: The special graph  $\Gamma$

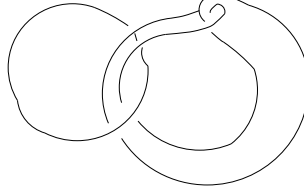


Figure 8: The Whitehead link

according to [?]. In fact we have :

$$J_L = t^{3L \cdot L} \sqrt{2} I(L),$$

where  $t = e^{\frac{2\pi\sqrt{-1}}{16}}$ .

4.3) **The cabling formula.** If  $L$  is a framed link  $\mathbf{k}$  a coloring then the following formula permits to compute  $J_{L,\mathbf{k}}$  in terms only of Jones polynomial of cablings of  $L$ . Specifically we have

$$J_{L,\mathbf{k}} = \sum_{\mathbf{j}=0}^{\mathbf{n}/2} (-1)^{\mathbf{j}} C_{\mathbf{n}-\mathbf{j}}^{\mathbf{j}} J_{L^{\mathbf{n}-2\mathbf{j}}} \quad (7)$$

where  $\mathbf{n} = \mathbf{k} - \mathbf{1}$ , and we set  $f(\mathbf{n}) = \prod_i f(n_i)$ ,  $\mathbf{m} < \mathbf{n}$  if  $m_i < n_i$  for all  $i$ 's etc. Also  $L^{\mathbf{c}}$  is the  $\mathbf{c}$ -cabling of  $L$  which consists in replacing the  $i^{th}$  component of  $L$  by  $c_i$  parallel copies.

In case when  $r = 4$  and the link  $L$  has two components  $K$  and  $H$ , both unknotted in  $\mathbf{R}^3$ , then the possible values for  $J_{L,\mathbf{k}}$  are

$$J_{L,(1,1)} = 1,$$

$$J_{L,(1,2)} = J_H = \sqrt{2},$$

$$J_{L,(2,1)} = J_K = \sqrt{2},$$

and using the 1-colored components removing lemma (see [?],p.511),

$$J_{L,(1,3)} = J_{H,3} = J_{H^2} - 1 = 1,$$

$$J_{L,(3,1)} = J_{K,3} = J_{K^2} - 1 = 1,$$

$$J_{L,(2,2)} = J_L$$

$$J_{L,(3,2)} = J_{K^2H} - J_H = J_{K^2H} - \sqrt{2},$$

$$J_{L,(2,3)} = J_{KH^2} - J_K = J_{KH^2} - \sqrt{2},$$

$$J_{L,(3,3)} = J_{K^2H^2} - J_{K^2} - J_{H^2} + 1 = J_{K^2H^2} - 3.$$

(4.3) **A surgical description of the cobordism  $X$ .** We come back now to the cobordism  $X$  from (2.4). We have a simple surgical description for  $X$  since both tori  $T_0$  and  $T_1$  are unknotted in  $S^3$ . We can choose for example the special graph  $\Gamma$  from picture 7. Taking into account that the intermediary link of  $\Gamma$  is trivial this time we see that

$$Z_j^i(\Gamma) = c^{-3} J_{\Gamma,(i,j)}, \text{ for } i, j \in \{1, 2, 3\}.$$

When properly interpreted  $J_{\Gamma,(i,j)}$  is  $J_{L,(i,j)}$  where  $L$  is the Whitehead link (see the picture 8). Since both components of  $L$  are unknotted we may apply the previous formulas. We compute first



$$I(L) = -\sqrt{2}, I(K^2H) = I(KH^2) = -2, I(K^2H^2) = 4.$$

Therefore up to a root of unity the morphism  $Z(X)$  is given by the matrix

$$\begin{pmatrix} 1 & \sqrt{2} & 1 \\ \sqrt{2} & \sqrt{2}(1 - \sqrt{-1}) & 2\sqrt{-2} - \sqrt{2} \\ 1 & 2 - 2\sqrt{-1} - \sqrt{2} & -3 - 4\sqrt{-1} \end{pmatrix}$$

whose determinant is  $2 - 13\sqrt{2} + (16 - 4\sqrt{2})\sqrt{-1}$ . Therefore the inductive limit  $\lim_{\rightarrow}(\mathbf{C}, Z(X))$  of iterates of the map  $Z(X)$  is isomorphic to  $\mathbf{C}^3$  and the claim of the theorem is proved. Notice that the change of the parametrization on intermediary boundaries amounts to multiply the matrix  $Z(X)$  by an invertible one which does not affect the limit.